UNIVERSITY OF ALBERTA
LIBRARY

Regulations Regarding Theses and Dissertations

Typescript copies of theses and dissertations for Master's and Doctor's degrees deposited in the University of Alberta Library, as the official Copy of the Faculty of Graduate Studies, may be consulted in the Reference Reading Room only.

A second copy is on deposit in the Department under whose supervision the work was done. Some Departments are willing to loan their copy to libraries, through the inter-library loan service of the University of Alberta Library.

These theses and dissertations are to be used only with due regard to the rights of the author. Written permission of the author and of the Department must be obtained through the University of Alberta Library when extended passages are copied. When permission has been granted, acknowledgement must appear in the published work.

This thesis or dissertation has been used in accordance with the above regulations by the persons listed below. The borrowing library is obligated to secure the signature of each user.

Please sign below:

Date	Signature	Institution

THE UNIVERSITY OF ALBERTA

APPLICATION OF THE N-QUANTUM APPROXIMATION TO THE PROBLEM OF BOUND STATE

by

MICHAEL MEIR BINDER

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS

EDMONTON, ALBERTA

FEBRUARY, 1968

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled APPLICATION OF THE N-QUANTUM APPROXIMATION TO THE PROBLEM OF BOUND STATE , submitted by Michael Meir Binder, in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

The bound state of two massive boson fields coupled together by a real scalar field is studied. Using the "N-quantum approximation", an integral equation for the "wave function" is obtained. The eigenvalues of the resulting equation are compared with the eigenvalues of the Schrodinger, Klein-Gordon, Bethe-Salpeter and Son-Sucher equations. When the binding energy is small compared with the masses of the fields, the coupling constant obtained from this equation is near to those obtained from the Klein-Gordon and Schrodinger equations, but is smaller than the coupling constant obtained from the Bethe-Salpeter and Son-Sucher equations.

[illegible]

ACKNOWLEDGEMENTS

I would like to thank Dr. M. Razavy, my supervisor, who initially proposed this project to me. His counsel, work and encouragement have been indispensable throughout the entire programme. Thanks are also due to Mr. R. G. Levers for reading and commenting on the manuscript.

Financial support from the Department of Physics, University of Alberta via a Graduate Teaching Assistantship is gratefully acknowledged.

CONTENT

LIST OF TABLES

LIST OF FIGURES

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

NOTATIONS

Throughout this work we shall use the metric $g^{\mu\upsilon} = \{1,-1,-1,-1\}$. We shall also use the following notations;

x, k, k' - 4-vector .

dx , dk , dk' , to be d^4x , d^4k , d^4k' respectively .

$\vec{p}$, $\vec{p}'$, $\vec{q}$, $\vec{r}$, $\vec{r}'$ - 3-vector, with p , p' , q , r , r' to be their magnitude respectively. Also

$$\Delta = \frac{\partial^2}{\partial t^2} - \nabla^2$$

$B = - E$ - The binding energy

$\lambda = G^2/4\pi$ - The coupling constant

$$t = \frac{1 + p^2 + q^2}{2pq}$$

$$\underline{t} = \frac{2(p^2+m_2^2)^{1/2}(q^2+m_2^2)^{1/2} + 1 - 2m_2^2}{2pq}$$

CHAPTER I. INTRODUCTION

The binding energy of a system has always been a quantity of great interest to the theorists in physics, since it could usually be measured experimentally. Thus, any theory attempting to explain the bound states could be immediately tested by experiment.

With the introduction of quantum mechanics, the binding energies of electrons in different systems such as hydrogen and helium could be explained reasonably well using the Schrodinger equation with the correct form of the potential[1]. Following this success, people have tried to extend this method to systems involving nuclear interactions. Hence, solutions of the Schrodinger equation using simple potentials such as the square well, exponential and gaussian potentials were used to explain nuclear bound state systems such as the deutron[2]. This method had two important drawbacks. First, to justify the use of the Schrodinger equation one had to assume that the bound particle had a nonrelativistic velocity; and second, one had no theoretical grounds to assume that any of these potentials actually represented the nuclear forces.

In 1935, Yukawa[3], in trying to formulate a consistent relativistic theory, had to assume the existence of a particle, now called a meson, having a rest mass between the electron and nucleon masses. He proposed a theory of nuclear forces which involves these mesons, and is referred to as the "meson theory of nuclear forces". The detection of such a particle in the cosmic radiation and the subsequent discovery that there exist several different mesons with different charges and rest masses, stimulated many attempts to fit these mesons into a consistent scheme of nuclear forces. These attempts, however, have not yet succeeded in reproducing quantitatively the known properties of nuclear forces. Thus, while there is still no general theory for the interaction of elementary particles, people have recently proposed different relativistic equations, based on different assumptions or methods, which still have no solid theoretical foundation. Two such equations are the Bethe-Salpeter equation[4] and a bound state equation proposed by Son and Sucher[5].

In a paper by Greenberg[6], a new method has been developed to obtain approximate solutions for the equations of motion of quantized fields. This tech-

nique, the "N-quantum approximation" is a covariant method of field quantization which is suitable for strong interaction physics, where the results of perturbative calculations are not reliable.

This work is an application of the "N-quantum approximation". We propose to determine the binding energy of two massive spinless fields Ψ_1 and Ψ_2 coupled together by a real scalar field Φ of unit mass. A similar problem for two spinor fields coupled together by a pseudoscalar meson field is discussed in a paper by Greenberg and Genolio[7]. We shall derive the equation for the "wave function" of the bound system and refer the reader to references (6) and (7) for a careful discussion about the assumptions, the nature, and the range of validity of the approximation.

For the sake of completeness, we outline here the essential features of the method. We assume that we are dealing with a field theory where there are irreducible sets of field operators. The irreducible sets of fields are the Heisenberg fields, which appear in the Lagrangian and Hamiltonian, but which have no

simple universal relation to the particles' states; and the in (out) fields, which do not appear in the Lagrangian or the Hamiltonian, but are the free fields describing the asymptotic in (out) states of freely moving particles. The basic idea of the "N-quantum approximation" is to exploit the expansion of the Heisenberg fields in terms of normal-ordered products of in-fields as was first given by Haag[8]. For an exact Heisenberg field of any non-trivial local field theory, this expansion never terminates; nonetheless, the expansion can terminate for an approximation to the exact Heisenberg field. Thus by substituting the truncated expansions for the Heisenberg fields into their equations of motion and using the fact that various normal ordered products of in-fields are independent, one obtains equations for the expansion coefficients. In general these equations are coupled sets of nonlinear singular integral equations. For the lowest order approximation, however, one obtains a linear equation.

The advantages of this approach are that the resultant equations are separable and relatively simple; that is to say, the time component can be separated out.

This is not the case for the Bethe-Salpeter equation. This method is also non-perturbative, and therefore it does not depend on an expansion in terms of the large coupling constant. Also higher order correction can be added in a straightforward manner as indicated in references (6) and (7).

In this work we determined the binding energies predicted by the lowest order approximation, and compared them with those obtained from the solutions of Schrodinger, Klein-Gordon, Bethe-Salpeter and Son-Sucher equations.

Chapter II. SCHRODINGER AND KLEIN-GORDON EQUATIONS IN MOMENTUM SPACE

We transform Schrodinger and Klein-Gordon equations into momentum space and obtain two linear integral equations which will be solved numerically in chapter IV.

II.1 Schrodinger Equation

Schrodinger equation for two particles of mass m_1 and m_2 in the relative coordinates is

$$- \frac{1}{2M} \nabla^2 \Psi(\vec{r}) + v(\vec{r})\Psi(\vec{r}) = E\Psi(\vec{r}) \tag{II-1}$$

with $\hbar^2 = 1$ and $M = m_1 m_2/(m_1+m_2)$.

We define the Fourier Transform of $u(\vec{p})$ and $v(\vec{p})$ by

$$\Psi(\vec{r}) = (2\pi)^{-\frac{3}{2}} \int e^{-i\vec{p}.\vec{r}} u(\vec{p}) d^3p = F[u] ,$$

and

$$v(\vec{r}) = (2\pi)^{-\frac{3}{2}} \int e^{-i\vec{p}.\vec{r}} v(\vec{p}) d^3p = F[v] .$$

Substituting these definitions into equation (II-1) we get

$$F\left[\frac{p^2}{2M}\,u(\vec{p})\right] + F\left[(2\pi)^{-\frac{3}{2}} \int v(\vec{p}-\vec{p}\,')u(\vec{p}\,')d^3p'\right] = EF[u(\vec{p})] \ ,$$

where we have used the convolution theorem

$$F[A]F[B] = F\left[(2\pi)^{-\frac{3}{2}} \int A(\vec{p}-\vec{p}\,')B(\vec{p}\,')d^3p'\right] .$$

Taking the inverse Fourier Transform and letting $E = -B$ where $B > 0$, we obtain

$$\left[\frac{p^2}{2M} + B\right]u(\vec{p}) = -(2\pi)^{-\frac{3}{2}} \int v(\vec{p}-\vec{p}\,')u(\vec{p}\,')d^3p' \ , \qquad \text{(II-2)}$$

where B is the binding energy of the system, and $u(\vec{p})$ is the bound state wave function. Since $v(\vec{p}-\vec{p}\,')$ is invariant under rotation we may write, with $x = \vec{p}.\vec{p}\,'/pp'$,

$$v(\vec{p}-\vec{p}\,') = \sum_{\ell=0}^{\infty} (2\ell+1)v_\ell(p,p')P_\ell(x) \ , \qquad \text{(II-3)}$$

where $P_\ell(x)$ are the Legendre polynomials. Using the identity

$$\int_{-1}^{1} P_{\ell'}(x)P_{\ell}(x)dx = \frac{2}{2\ell+1}\,\delta_{\ell\ell'} \quad , \tag{II-4}$$

we obtain

$$v_{\ell}(p,p') = \frac{1}{2}\int_{-1}^{1} v(\vec{p}-\vec{p}\,')P_{\ell}(x)dx \quad . \tag{II-5}$$

We now set

$$u(\vec{p}) = \Sigma\, u_{\ell}(p)Y_{\ell m}(\hat{p}) \tag{II-6}$$

where $Y_{\ell m}(\hat{p})$ are the usual spherical harmonics. Substituting equations (II-3) and (II-6) into equation (II-2), and using

$$\int P_{\ell'}(\hat{p}.\hat{p}\,')Y_{\ell m}(\hat{p}\,')d\hat{p}\,' = \frac{4\pi}{2\ell+1}\,Y_{\ell m}(\hat{p})\,\delta_{\ell\ell'} \tag{II-7}$$

we get the reduced equation

$$[\frac{p^2}{2M} + B]u_{\ell}(p) = -(\frac{2}{\pi})^{\frac{1}{2}}\int_{0}^{\infty} v_{\ell}(p,p')u_{\ell}(p')p'^2dp' \quad . \tag{II-8}$$

For spherically symmetric potentials,

$$v(\vec{p}) = (2\pi)^{-\frac{3}{2}}\int e^{i\vec{r}.\vec{p}}\, v(\vec{r})d^3r$$

$$= (2\pi)^{-\frac{1}{2}} \int_0^\infty r^2 v(r)dr \int_{-1}^{1} e^{ipry}\, dy$$

$$= (\tfrac{2}{\pi})^{\frac{1}{2}} \int_0^\infty p^{-1}\sin(pr)v(r)rdr \quad .$$

Therefore,

$$v(\vec{p}-\vec{p}\,') = (\tfrac{2}{\pi})^{\frac{1}{2}} \int_0^\infty \frac{\sin|\vec{p}-\vec{p}\,'|\, r}{|\vec{p}-\vec{p}\,'|} v(r)rdr \quad . \qquad \text{(II-9)}$$

Using the identity

$$\frac{\sin|\vec{p}-\vec{p}\,'|\, r}{|\vec{p}-\vec{p}\,'|} = \pi \sum_{m=0}^{\infty} (pp')^{-\frac{1}{2}}(m+1/2)J_{m+1/2}(pr)J_{m+1/2}(p'r) \cdot P_m(x) \quad ,$$

where $J_{m+1/2}(z)$ is the Bessel function, and equations (II-9) and (II-4), we can write equation (II-5) as

$$v_\ell(p,p') = (\tfrac{\pi}{2})^{\frac{1}{2}} \int_0^\infty (pp')^{-\frac{1}{2}} J_{\ell+1/2}(pr)J_{\ell+1/2}(p'r)v(r)rdr \quad . \qquad \text{(II-10)}$$

Equation (II-8), with $v_\ell(p,p')$ given by equation (II-10), is our Schrodinger equation in momentum space and we shall solve it later on for different potentials $v(r)$.

II.2 Klein-Gordon Equation

We write the Klein-Gordon equation for a particle of mass m, in a central field of force $v(\vec{r})$, as

$$[\Delta + m^2]\Psi(\vec{r},t) = -2mv(\vec{r})\Psi(\vec{r},t) \quad , \qquad \text{(II-11)}$$

where the factor 2m is necessary to give the correct non-relativistic limit of equation (II-2). Let

$$\Psi(\vec{r},t) = \psi(\vec{r})e^{-iEt} \quad ,$$

where $E = m-B$ is the bound state energy. Then equation (II-11) becomes

$$\nabla^2\psi(\vec{r}) + (E^2-m^2)\psi(\vec{r}) - 2mv(\vec{r})\psi(\vec{r}) = 0 \ ,$$

or

$$-\frac{1}{2m}\nabla^2\psi(\vec{r}) + v(\vec{r})\psi(\vec{r}) = -\frac{B(2m-B)}{2m}\psi(\vec{r}) \quad . \qquad \text{(II-12)}$$

Note that equation (II-12) is identical with equation (II-1) provided we replace in equation (II-1) $-E = B$ by $B(2m-B)/2m$ and $2M$ by $2m$. Hence, the analogue

of equation (II-8) is now

$$\left[\frac{p^2}{2m} + \frac{B(2m-B)}{2m}\right]u_\ell(p) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty v_\ell(p,p')u_\ell(p')p'^2dp' \tag{II-13}$$

where $v_\ell(p,p')$ is still given by equation (II-10) .

II.3 The Yukawa Potential

Consider the Yukawa potential

$$v(r) = -\lambda\frac{e^{-\mu r}}{\mu r},$$

where $\lambda = \frac{G^2}{4\pi}$ is the coupling constant and we choose μ to be unity. Equation (II-10) now becomes

$$\begin{aligned} v_\ell(p,p') &= -\lambda\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \int_0^\infty (pp')^{-\frac{1}{2}} J_{\ell+1/2}(pr)J_{\ell+1/2}(p'r)e^{-r}dr \\ &= -(2\pi)^{-\frac{1}{2}}\frac{\lambda}{pp'}\, Q_\ell\left(\frac{1+p^2+p'^2}{2pp'}\right) \end{aligned}$$

where $Q_\ell(z)$ are the Legendre functions of the second kind. Thus equations (II-8) and (II-13) now become

$$[\frac{p^2}{2M} + B]u_\ell(p) = \frac{\lambda}{\pi} \int_0^\infty Q_\ell(t)u_\ell(q)q\,\frac{dq}{p} \quad , \tag{II-14}$$

and

$$[\frac{p^2}{2m} + \frac{B(2m-B)}{2m}]u_\ell(p) = \frac{\lambda}{\pi} \int_0^\infty Q_\ell(t)u_\ell(q)q\,\frac{dq}{p} \tag{II-15}$$

respectively, where

$$t = \frac{1+p^2+q^2}{2pq} \quad .$$

Chapter III. NEW RELATIVISTIC EQUATION FOR BOUND STATE

In this chapter we shall derive a relativistic equation using the lowest-order "N-quantum approximation". This equation reduces to the Schrodinger and Klein-Gordon equations in the proper limits.

III.1 Equations of Motion

Let the Hamiltonian for two scalar fields Ψ_1 and Ψ_2 of masses m_1 and m_2 respectively, which interact with the exchange of a scalar meson Φ of unit mass, be:

$$H = \int \bar{H}\, d\tau$$

$$= \int \sum_{i=1}^{2} [\ \pi_i(x)\bar{\pi}_i(x) + (\nabla\Psi_i)(\nabla\bar{\Psi}_i) + m_i^2\Psi_i(x)\bar{\Psi}_i(x)]d\tau$$

$$+ \frac{1}{2}\int[\rho^2(x) + \Phi^2(x) + (\nabla\Phi)^2]d\tau + G\int \sum_{i=1}^{2} m_i\ \Phi\{\Psi_i,\bar{\Psi}_i\}d\tau, \tag{III-1}$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

where $\bar{H}$ is the Hamiltonian density. From Hamilton's canonical equations, it follows that

$$\frac{\partial \Psi_i}{\partial t} = \frac{\partial \bar{H}}{\partial \pi_i} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial \bar{H}}{\partial\left(\frac{\partial \pi_i}{\partial x_j}\right)} = \bar{\pi}_i \ ,$$

and

$$-\frac{\partial \bar{\pi}_i}{\partial t} = \frac{\partial \bar{H}}{\partial \bar{\Psi}_i} - \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \frac{\partial \bar{H}}{\partial\left(\frac{\partial \bar{\Psi}_i}{\partial x_j}\right)}$$

$$= m_i^2 \Psi_i + 2Gm_i \Phi \Psi_i - \nabla^2 \Psi_i \quad .$$

Therefore,

$$-\frac{\partial^2 \Psi_i}{\partial t^2} = -\frac{\partial \bar{\pi}_i}{\partial t} = m_i^2 \Psi_i + 2Gm_i \Phi \Psi_i - \nabla^2 \Psi_i$$

or,

$$[\ \Delta + m_i^2\ ]\ \Psi_i(x) = -\ 2Gm_i \Phi \Psi_i \quad . \tag{III-2}$$

Similarly, the equation of motion of the meson field turns out to be

$$[\ \Delta + 1\ ]\ \Phi(x) = -\ G \sum_{i=1}^{2} m_i \{\Psi_i , \bar{\Psi}_i\} \quad . \tag{III-3}$$

Equations (III-2) and (III-3) are the equations of motion of the fields. Note however, that equation (III-2) is not symmetric in Φ and Ψ_i , and since, from equation (III-3), Φ will depend in general upon Ψ_i and $\bar{\Psi}_i$, we would like to symmetrize equation (III-2) to prevent any ambiguity. Since Φ and Ψ_i commute, we can rewrite equation (III-2) as

$$[\ \Delta + m_i^2\ ]\ \Psi_i(x) = -\ Gm_i \{\Phi , \Psi_i\} \quad . \tag{III-4}$$

Equations (III-3) and (III-4) can be

transformed into momentum space by defining

$$\Phi(x) = \int \Phi'(k)e^{-ikx}\,dk \quad ,$$

$$\Psi_i(x) = \int \Psi_i'(k)e^{-ikx}\,dk$$

and

$$\bar{\Psi}_i(x) = \int \bar{\Psi}_i'(k)e^{-ikx}\,dk \quad .$$

Equation (III-4) then becomes:

$$\int \Psi_i(k)(m_i^2-k^2)e^{-ikx}dk = -\,Gm_i\int\{\Phi(k_1),\Psi_i(k_2)\}e^{-i(k_1+k_2)x} \cdot dk_1dk_2$$

where we have dropped the primes on Φ and Ψ_i . Multiplying both sides of the above equation by $e^{ik'x}dx$, integrating over x, and using the identity

$$\int e^{i(k-k')x}\,dx = (2\pi)^4\delta(k-k') \quad ;$$

we get

$$(m_i^2-k^2)\Psi_i(k) = -\,Gm_i\int\delta(k-k_1-k_2)\{\Phi(k_1),\Psi_i(k_2)\}dk_1dk_2 \quad . \tag{III-5}$$

Similarly equation (III-3) becomes

$$(1-k^2)\Phi(k) =$$

$$- G \sum_{i=1}^{2} m_i \int \delta(k-k_1-k_2)\{\Psi_i(k_1),\bar{\Psi}_i(k_2)\}dk_1 dk_2 \quad . \quad \text{(III-6)}$$

Equations (III-5) and (III-6) are the equations of motion of the Heisenberg fields in momentum space.

III.2 The "N-Quantum Approximation"

Now we want to write down the truncated expansion for the Heisenberg fields in terms of the in-fields. Since Ψ_1 , Ψ_2 , and Φ represent particles which are stable; the set of in-fields will include Ψ_1^{in} , Ψ_2^{in} , and Φ^{in} . Each stable bound state will also lead to an in-field, and since we are dealing with the lowest-order "N-quantum approximation", which neglects any radiative transition, all our bound states are stable. Hence, the set of in-fields will include $D_{n\ell}^{in}$, the bound state in-fields, where n is the principle quantum number and ℓ is the angular momentum.

To obtain the lowest-order "N-quantum approximation" we use following Greenberg's work[6,7], the following truncated expansions;

$$\Phi(k) = \Phi^{in}(k)\delta_1(k)$$

$$+ \int dk_1 dk_2 \delta(k-k_1-k_2) \sum_{i=1}^{2} g_i(k_1,k_2) : \Psi_i^{in}(k_1)\delta_{m_i}(k_1)\bar{\Psi}_i^{in}(k_2)\delta_{m_i}(k_2): \ ,$$

and (III-7)

$$\Psi_1(k) = \Psi_1^{in}(k)\delta_{m_1}(k) + \int dk_1 dk_2 \delta(k-k_1-k_2) \sum_{n,\ell} f_{n\ell}(k_1,k_2)$$

$$\cdot \quad :\Psi_2^{in}(k_1)\delta_{m_2}(k_1) D_{n\ell}^{in}(k_2)\delta_{d_{n\ell}}(k_2): \ , \qquad \text{(III-8)}$$

where $f_{n\ell}(k_1,k_2)$ and $g_i(k_1,k_2)$ are the vertex functions which are related, in the weak coupling limit, to the matrix elements

$\langle \Psi_2^{in} | \Psi_1 | D_n^{in} \rangle$ and $\langle \Psi_i^{in} | \Phi | \bar{\Psi}_i^{in} \rangle$ respectively.

Also $\delta_m(k) \equiv \delta(m^2-k^2)$, $D_{n\ell}^{in}$ are the bound states' in-fields, and $d_{n\ell} = m_1+m_2-B_{n\ell}$ are the masses of the bound states with the binding energies $B_{n\ell}$. There are similar expansions for Ψ_2 and $\bar{\Psi}_i$ but we shall not need them here. In what follows we shall often abbreviate an expression such as $\bar{\Psi}_2^{in}(k_1)\delta_{m_2}(k_1)D_{n\ell}^{in}(k_2)\delta_{d_{n\ell}}(k_2)$ to

$\bar{\Psi}_2(1)D_{n\ell}(2)$ when there is no danger of an ambiguity. Substitution of equations (III-7) and (III-8) into equation (III-5) yields:

$$\int dk_1 dk_2 (m_1^2 - k^2)\delta(k-k_1-k_2) \sum_{n,\ell} f_{n\ell}(k_1,k_2) :\bar{\Psi}_2(1)D_{n\ell}(2):$$

$$= - Gm_1 \int dk_1 dk_2 \delta(k-k_1-k_2) \{ [\Phi(1) + \int dk_3 dk_4 \delta(k_1-k_3-k_4)$$

$$\cdot \sum_{i=1}^{2} g_i(k_3,k_4) :\Psi_i(3)\bar{\Psi}_i(4):], [\Psi_1(2) + \int dk_5 dk_6 \delta(k_2-k_5-k_6)$$

$$\cdot \quad f_{n\ell}(k_5,k_6) :\bar{\Psi}_2(5)D_{n\ell}(6):] \} \quad . \qquad \text{(III-9)}$$

Since normal products of in-fields are independent of each other, we shall try to equate the coefficient of $:\bar{\Psi}_2(1)D_{n\ell}(2):$ on the left hand side of equation (III-9), with those coefficients of the normal products, on the right hand side of equation (III-9), which are of the same order. Since we are interested in the lowest-order approximation we shall drop any higher terms of the normal products. Hence, the only term to survive, from the right hand side of equation (III-9) is $\{:\Psi_1(3)\bar{\Psi}_1(4) + \Psi_2(3)\bar{\Psi}_2(4):, :\bar{\Psi}_2(5)D_{n\ell}(6):\}$. The term $\{:\Psi_1(3)\bar{\Psi}_1(4):, :\bar{\Psi}_2(5)D_{n\ell}(6):\}$ however, cannot be further contracted to lower order and thus, in the lowest-order

approximation, it will be ignored. Therefore, we are left with

$$
\{:\Psi_2(3)\bar{\Psi}_2(4):, :\bar{\Psi}_2(5)D_{n\ell}(6):\}
$$

$$
= \,:\Psi_2(3)\bar{\Psi}_2(4):\bar{\Psi}_2(5)D_{n\ell}(6): + :\bar{\Psi}_2(5)D_{n\ell}(6):\Psi_2(3)\bar{\Psi}_2(4):
$$

$$
= \overbracket{\Psi_2(3)\bar{\Psi}_2}(5):\bar{\Psi}_2(4)D_{n\ell}(6): + \overbracket{\bar{\Psi}_2(5)\Psi_2}(3):\bar{\Psi}_2(4)D_{n\ell}(6): \quad . \tag{III-10}
$$

To find the explicit form of the contraction $\overbracket{\Psi_2(3)\bar{\Psi}_2}(5)$ and $\overbracket{\bar{\Psi}_2(5)\Psi_2}(3)$ we use the commutation relation;

$$
[\Psi^{in}(k_1)\delta_m(k_1), \bar{\Psi}^{in}(k_2)\delta_m(k_2)] = (2\pi)^{-3}\epsilon(k_1)\delta_m(k_1)\delta(k_1+k_2) \quad , \tag{III-11}
$$

where $\epsilon(k) = \begin{cases} +1 & \text{for } k^0 > 0 \\ -1 & \text{for } k^0 < 0 \end{cases}$.

Taking the vacuum amplitude we get

$$
\langle \Psi^{in}(k_1)\delta_m(k_1)\bar{\Psi}^{in}(k_2)\delta_m(k_2)\rangle_o = (2\pi)^{-3}\theta(k_1)\delta_m(k_1)\delta(k_1+k_2) \quad , \tag{III-12}
$$

where $\theta(k) = \begin{cases} 1 & \text{for } k^0 > 0 \\ 0 & \text{for } k^0 < 0 \end{cases}$. Then,

$$\langle \bar{\Psi}^{in}(k_2)\delta_m(k_2)\Psi^{in}(k_1)\delta_m(k_1)\rangle_0 = (2\pi)^{-3}\theta(k_2)\delta_m(k_2)\delta(k_1+k_2) \tag{III-13}$$

and, as can be verified directly, equations (III-12) and (III-13) satisfy the commutation relation (III-11). Therefore, using equations (III-12) and (III-13) we can write equation (III-10) as

$$\{:\Psi_2(3)\bar{\Psi}_2(4):,:\bar{\Psi}_2(5)D_{n\ell}(6):\}$$

$$= (2\pi)^{-3}\delta_{m_2}(3)\delta(k_3+k_5)[\theta(k_3)+\theta(-k_3)]:\bar{\Psi}_2(4)D_{n\ell}(6)$$

$$= (2\pi)^{-3}\delta_{m_2}(3)\delta(k_3+k_5):\bar{\Psi}_2(4)D_{n\ell}(6): \quad . \tag{III-14}$$

Equation (III-9) now becomes

$$\int dk_1 dk_2(m_1^2-k^2)\delta(k-k_1-k_2)\sum_{n,\ell} f_{n\ell}(k_1,k_2):\bar{\Psi}_2(1)D_{n\ell}(2):$$

$$= -\frac{Gm_1}{(2\pi)^3}\int dk_1\ldots dk_6\delta(k-k_1-k_2)\delta(k_1-k_3-k_4)\delta(k_2-k_5-k_6)$$

$$\cdot \sum_{n,\ell} f_{n\ell}(k_5,k_6)g_2(k_3,k_4):\Psi_2(4)D_{n\ell}(6):$$

$$= - \frac{Gm_1}{(2\pi)^3}\int dk_1 dk_2 dk_3 \delta(k-k_1-k_2) \sum_{n,\ell} f_{n\ell}(-k_3,k_2+k_3) g_2(k_3,k_1-k_3)$$

$$\cdot \quad \delta_{m_2}(k_3) : \bar{\Psi}_2(1-3) D_{n\ell}(2+3)$$

$$= - \frac{Gm_1}{(2\pi)^3}\int dk_1 dk_2 dk_3 \delta(k-k_1-k_2) \sum_{n,\ell} f_{n\ell}(k_3,k_2) g_2(-k_3,k_1)$$

$$\cdot \quad \delta_{m_2}(k_3) : \bar{\Psi}_2(1) D_{n\ell}(2) : \quad , \tag{III-15}$$

where we used the fact that we can interchange the order of integration and thus, relable the dummy variables k_1-k_3 as k_1 , k_2+k_3 as k_2 , and k_3 as $-k_3$. Since equation (III-15) is true for all k , n , and ℓ we then have;

$$[m_1^2-(k_1+k_2)^2] f_{n\ell}(k_1,k_2) = - \frac{Gm_1}{(2\pi)^3}\int dk_3 f_{n\ell}(k_3,k_2) g_2(-k_3,k_1) \delta_{m_2}(k_3) \tag{III-16}$$

with the constrains; $k_1^2 = m_2^2$ and $k_2^2 = d_{n\ell}^2$. To calculate $g_2(-k_3,k_1)$ explicitly, we substitute equations (III-7) and (III-8) into equation (III-6), and retain only the first term on the right hand side of the expansions for Ψ_i and $\bar{\Psi}_i$, which amount to just the Born approximation. Thus,

$$\int dk_1 dk_2 (1-k^2)\delta(k-k_1-k_2) \sum_{i=1}^{2} g_i(k_1,k_2) :\Psi_i(1)\bar{\Psi}_i(2):$$

$$= - G \sum_{i=1}^{2} m_i \int dk_1 dk_2 \delta(k-k_1-k_2) :\{\Psi_i(1),\bar{\Psi}_i(2)\}: \quad . \qquad \text{(III-17)}$$

Let us write Ψ_i^{in} and $\bar{\Psi}_i^{in}$ in terms of the creation and annhilation operators, Ψ_i^+ and Ψ_i^- respectively, as

$$\Psi_i^{in}(k_1) = N[\Psi_i^+(1) + \Psi_i^-(1)] \quad ,$$

and

$$\bar{\Psi}_i^{in}(k_2) = N[\Psi_i^-(2) + \Psi_i^+(2)]$$

where N is the normalization constant. Taking the vacuum state expectation value of both sides of equation (III-17), and using the properties of the creation and annhilation operators, and the orthogonality property of states; we get

$$[1 - (k_1+k_2)^2] g_i(k_1,k_2) = - 2Gm_i$$

or

$$g_i(k_1,k_2) = - \frac{2Gm_i}{1 - (k_1+k_2)^2} \quad . \qquad \text{(III-18)}$$

Thus, equation (III-16) becomes

$$[m_1^2 - (k_1+k_2)^2]f_{n\ell}(k_1,k_2) = \frac{2G^2m_1m_2}{(2\pi)^3}\int dk\delta_{m_2}(k)\frac{f_{n\ell}(k,k_2)}{1-(k_1-k)^2} \quad . \tag{III-19}$$

We require that equation (III-19) should have the correct non-relativistic limit. This can be achieved by choosing $k_2^0 > 0$ and $k_1^0 < 0$. Hence, we rewrite equation (III-19) as

$$[m_1^2 - (k_1+k_2)^2]f_{n\ell}(k_1,k_2) = \frac{4G^2m_1m_2}{(2\pi)^3}\int dk\delta_{m_2}(k)\theta(-k)\frac{f_{n\ell}(k,k_2)}{1-(k_1-k)^2} \quad . \tag{III-20}$$

To see that the factor 2 on the right hand side of equation (III-20) is the correct one, we note that if we have used the unsymmetrized form of equation (III-4), namely, equation (III-2); we would have obtained equation (III-20) directly.

The conditions

$$k_1^2 = m_2^2 \quad , \quad k_1^0 < 0 \quad ; \quad k_2^2 = d_{n\ell}^2 \quad , \quad k_2^0 > 0$$

suggest that we choose our coordinates as:

$$k_1 = \{-(p^2+m_2^2)^{\frac{1}{2}};\ \vec{p}\} \quad ,$$

$$k_2 = \{\ d_{n\ell}\ ;\ \vec{0}\ \} \quad ,$$

$$k = \{\ -(q^2+m_2^2)^{\frac{1}{2}}\ ;\ \vec{q}\ \} \quad .$$

Then, equation (III-20) reduces to

$$[m_1^2 - m_2^2 - d_{n\ell}^2 + 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}}]f_{n\ell}(\vec{p})$$

$$= \frac{4G^2 m_1 m_2}{(2\pi)^3} \int dk\, \delta_{m_2}(k)\theta(-k) \frac{f_{n\ell}(k)}{1 - m_2^2 + 2k_1 k - m_2^2} \quad .$$

With

$$2k_1 k = 2(k_1^0 k^0 - \vec{k}_1 . \vec{k}) = 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}} - 2\vec{k}_1 . \vec{q}$$

and

$$\delta_{m_2}(k) = \delta(m_2^2 - k^2) = \delta(m_2^2 - k^{0^2} + \vec{k}^2)$$

$$= \delta\{[(q^2+m_2^2)^{\frac{1}{2}} - k^0][(q^2+m_2^2)^{\frac{1}{2}} + k^0]\}$$

$$= \frac{1}{2}(q^2+m_2^2)^{-\frac{1}{2}}\{\delta[k^0 - (q^2+m_2^2)^{\frac{1}{2}}] + \delta[k^0 + (q^2+m_2^2)^{\frac{1}{2}}]\ \}$$

we have

$$[d_{n\ell}^2 + m_2^2 - m_1^2 - 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}}]f_{n\ell}(\vec{p})$$

$$= \frac{-2G^2 m_1 m_2}{(2\pi)^3} \int d^3q (q^2+m_2^2)^{-\frac{1}{2}} \int_{-\infty}^{0} dk^0 \frac{\delta[k^0 + (q^2+m_2^2)^{\frac{1}{2}}]\ f_{n\ell}(\vec{q})}{1 - 2m_2^2 + 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}} - 2\vec{p}.\vec{q}}$$

$$= - \frac{2G^2 m_1 m_2}{(2\pi)^3} \int d^3q \frac{(q^2+m_2^2)^{-\frac{1}{2}}\ f_{n\ell}(\vec{q})}{1 - 2m_2^2 + 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}} - 2\vec{p}.\vec{q}} \quad . \tag{III-21}$$

Let

$$D \equiv 1 - 2m_2^2 + 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}} - 2\vec{p}.\vec{q}$$

$$= 2pq[\ \frac{1 - 2m_2^2 + 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}}}{2pq} - \frac{\vec{p}.\vec{q}}{pq}\ ]$$

$$= 2pq(\underline{t}-s)$$

where

$$\underline{t} \equiv \frac{1 - 2m_2^2 + 2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}}}{2pq} \quad ,$$

and

$$s \equiv \frac{\vec{p}\cdot\vec{q}}{pq} \leq 1 \quad .$$

Using the identity

$$\frac{1}{\underline{t} - s} = \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(s) Q_\ell(\underline{t}) \qquad |s| \leq 1$$

we get

$$\frac{1}{D} = \frac{1}{2pq} \sum_{\ell=0}^{\infty} (2\ell+1) P_\ell(s) Q_\ell(\underline{t}) \quad .$$

Hence, equation (III-21) reduces to

$$[d_{n\ell}^2 + m_2^2 - m_1^2 - 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}}] f_{n\ell}(\vec{p})$$

$$= - \frac{G^2 m_1 m_2}{(2\pi)^3} \sum_{\ell=0}^{\infty} P_\ell(s)(2\ell+1) \int d^3q \, \frac{Q_\ell(\underline{t}) \; f_{n\ell}(\vec{q})}{pq(q^2+m_2^2)^{\frac{1}{2}}} \quad .$$

Let

$$f_{n\ell}(\vec{p}) = f'_{n\ell}(p) Y_{\ell m}(\hat{p}) \quad ;$$

using equations (II-4) and (II-7) we finally obtain

$$[d_{n\ell}^2 + m_2^2 - m_1^2 - 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}}] f_{n\ell}(p) = - \frac{G^2 m_1 m_2}{2\pi^2} \int_0^{\infty} dq \frac{q Q_\ell(\underline{t}) f_{n\ell}(q)}{p(q^2+m_2^2)^{\frac{1}{2}}} \; , \tag{III-22}$$

where we have dropped the prime on f . Equation (III-22) is our relativistic equation and we shall solve it numerically later on.

III.3 Reduction to Schrodinger Equation

Consider equation (III-22) in the non-relativistic limit when the masses of the particles are much greater than their momenta and binding energies. Then,

$$\underline{t} = \frac{2(p^2+m_2^2)^{\frac{1}{2}}(q^2+m_2^2)^{\frac{1}{2}} + 1 - 2m_2^2}{2pq}$$

$$\approx \frac{1}{2pq}\left[\ 2m_2^2\left(1 + \frac{p^2}{2m_2^2}\right)\left(1 + \frac{q^2}{2m_2^2}\right) + 1 - 2m_2^2\ \right]$$

$$\approx \frac{1}{2pq}\left[\ 2m_2^2\left(1 + \frac{p^2}{2m_2^2} + \frac{q^2}{2m_2^2}\right) + 1 - 2m_2^2\ \right]$$

$$= \frac{p^2 + q^2 + 1}{2pq} = t \quad ,$$

$$(q^2 + m_2^2)^{-\frac{1}{2}} \approx \frac{1}{m_2}\left(1 - \frac{q^2}{2m_2^2}\right) \approx \frac{1}{m_2} \quad ,$$

and

$$d_{n\ell}^2 + m_2^2 - m_1^2 - 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}}$$

$$= (m_1+m_2-B_{n\ell})^2 + m_2^2 - m_1^2 - 2(m_1+m_2-B_{n\ell})(p^2+m_2^2)^{\frac{1}{2}}$$

$$\approx 2m_2^2 + 2m_1m_2 - 2B_{n\ell}m_1 - 2B_{n\ell}m_2 + B_{n\ell}^2 - (m_1+m_2-B_{n\ell})(2m_2 + \frac{p^2}{m_2})$$

$$\approx - 2m_1\left(\frac{p^2}{2} \frac{m_1+m_2}{m_1m_2} + B_{n\ell} \right) = - 2m_1\left(\frac{p^2}{2M} + B_{n\ell}\right) .$$

Thus, in the non-relativistic limit, equation (III-22) reduces to

$$\left[\frac{p^2}{2M} + B_{n\ell}\right]f_{n\ell}(p) = \frac{G^2}{4\pi} \frac{1}{\pi} \int_0^\infty dq \frac{q}{p} Q_\ell(t) f_{n\ell}(q) ,$$

which is just the Schrodinger equation (II-14) for the Yukawa potential with $\lambda = G^2/4\pi$.

III.4 Reduction to the Klein-Gordon Equation

Consider equation (III-22) in the limit of $m_2 \longrightarrow \infty$. Then,

$$\underline{t} \approx \frac{1 + p^2 + q^2}{2pq} = t ,$$

$$(q^2 + m_2^2)^{-\frac{1}{2}} \approx \frac{1}{m_2} ,$$

and

$$d_{n\ell}^2 + m_2^2 - m_1^2 - 2d_{n\ell}(p^2+m_2^2)^{\frac{1}{2}} \approx - 2m_1 B_{n\ell} + B_{n\ell}^2 - p^2$$

$$\approx - [B_{n\ell}(2m_1 - B_{n\ell}) + p^2] \quad .$$

Thus equation (III-22) becomes,

$$[\frac{p^2}{2m_1} + \frac{B_{n\ell}(2m_1-B_{n\ell})}{2m_1}]f_{n\ell}(p) = \frac{G^2}{4\pi}\frac{1}{\pi}\int_0^\infty dq \, \frac{q}{p} \, Q_\ell(t) \, f_{n\ell}(q) \ ,$$

which is just the Klein-Gordon equation, (II-22), with $\lambda = G^2/4\pi$.

III.5 Other Relativistic Equations

The Bethe-Salpeter equation for bound states of two identical scalar particles with mass m mediated by a finite mass meson, is given by

$$[(\tfrac{1}{2}K+k)^2 - m^2][(\tfrac{1}{2}K-k)^2 - m^2]\varphi_K(k)$$

$$= - \frac{4im^2G^2}{(2\pi)^4}\int d^4k' \, \frac{\varphi_K(k')}{(k-k')^2 + \mu^2} \ , \qquad \text{(III-24)}^5$$

where μ is the meson mass, G is the coupling constant,

K is the four-momentum of the bound state in the centre-of-mass system and k is the relative momentum.

Son and Sucher[5] introduced another relativistic equation for a bound particle of mass m_b ;

$$[2E(\vec{p}) - m_b]\varphi(\vec{p}) = - \int \underline{v}(\vec{p},\vec{p}\,')\varphi(\vec{p}\,')d^3p' \quad , \qquad \text{(III-25)}$$

where

$$\underline{v}(\vec{p},\vec{p}\ ) \equiv - \frac{G^2}{(2\pi)^3} \frac{m}{E(\vec{p})} \frac{1}{[(\vec{p}-\vec{p}\,')^2 + \mu^2]} \frac{m}{E(\vec{p}\,')}$$

and $E(\vec{p}) = (p^2 + m^2)^{\frac{1}{2}}$. Both equations (III-24) and (III-25) have the same non-relativistic limit, the equal-mass Schrodinger equation for a Yukawa potential $v(r)$:

$$v(r) = - \frac{G^2}{4\pi} \frac{e^{-\mu r}}{r} \quad .$$

We shall use the numerical solutions of equations (III-24) and (III-25) given by Schwartz[4] and Son-Sucher respectively, to compare with our results.

Chapter IV. METHOD OF SOLUTION

Numerical solutions for the eigenvalues of equations (II-14) and (II-15) are obtained using Bateman's[9] method.

IV.1 Bateman's Method

Consider the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_a^b K(x,y)\phi(y)dy \quad . \tag{IV-1}$$

To obtain the eigenvalues λ , of equation (IV-1), Bateman developed the following procedure. Define

$$T(x_\ell,y_m) = \int_a^b K(x_\ell,z)K(z,y_m)dz \quad , \tag{IV-2}$$

where $\ell,m = 1,2,\ldots\ldots n$, n is a non-zero positive integer, and x_ℓ and y_m are given numbers in the interval $a \leq x_\ell, y_m \leq b$, such that T and K are completely defined in this interval. Also define

$$A_{\ell m} = - K(x_\ell,y_m) + \lambda\, T(x_\ell,y_m) \quad .$$

Then, the eigenvalues of equation (IV-1) are given approximately by

$$\det\ (\ A_{\ell m}\ ) = 0 \quad ;$$

the degree of approximation improves as n increases. For the proof of the above procedure and for a discussion about its range of validity, we refer the reader to reference (9).

IV.2 Illustration of the Method

Generally, in order that the eigenvalues of an integral equation, such as equation (IV-1), be real we must have $K(x,y) = K(y,x)$. Hence, we write the Schrodinger equation (II-8) in a symmetric form with, $m_1 = m_2 = m$, as

$$X_\ell(p) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} m \int_0^\infty dp' X_\ell(p') \frac{v_\ell(p,p')pp'}{(p^2+mB)^{\frac{1}{2}}(p'^2+mB)^{\frac{1}{2}}} , \tag{IV-3}$$

where $X_\ell(p) = p(p^2 + mB)^{\frac{1}{2}} u_\ell(p)$. Note that the kernel of equation (IV-3) is symmetric since $v_\ell(p,p')$

is symmetric, and also that equation (IV-3) is of the same form as equation (IV-1). It turns out that we do not have to symmetrize our equation since in this method of approximation all the unsymmetric parts of the kernel drop out. Nevertheless we shall retain the symmetric form of our equations.

To illustrate the method of approximation, we shall solve equation (IV-3), with $\ell = 0$, for two potentials.

(i) Delta-function potential

Suppose we are given the potential $v(r) = -\lambda\delta(r-c)$, where c is a constant. Then, using equation (II-10), for $\ell=0$, we get

$$v_0(p,p') = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lambda\,\frac{\sin(pc)\sin(p'c)}{pp'} \quad .$$

Thus, the kernel of equation (IV-3) becomes

$$K(x,y) = \frac{2m}{\pi}\,\frac{\sin(xc)\sin(yc)}{(x^2+\omega^2)^{\frac{1}{2}}(y^2+\omega^2)^{\frac{1}{2}}} \quad ,$$

where $\omega^2 = mB$. Applying Bateman's method to determine the eigenvalues λ we get

$$\lambda = \frac{2\omega}{(1 - e^{-2\omega c})m} \quad ,$$

which is the exact solution for this potential, as one can easily verify by solving the Schrodinger equation directly. From this example we can conclude that any potential $v(r)$ for which $v_\ell(p,p')$ can be written in the separable form $v_\ell(p,p') = A_\ell(p)B_\ell(p')$, in this method of approximation, will yield exact solution.

(ii) Exponential potential

Consider the exponential potential $v(r) = -\lambda\mu^2 e^{-\mu r}$. Equation (II-10) then yields

$$v_0(p,p') = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\lambda\frac{2\mu^3}{[\mu^2+(p-p')^2][\mu^2+(p+p')^2]} \quad ,$$

and the kernel of equation (IV-3) is

$$K(x,y) = \frac{4m\mu^3}{\pi}\frac{xy}{[\mu^2+(x-y)^2][\mu^2+(x+y)^2]}(x^2+mB)^{-\frac{1}{2}}(y^2+mB)^{-\frac{1}{2}} \quad .$$

To find the eigenvalues λ , using Bateman's method, we must solve $\det(A_{\ell m}) = 0$ or

$$\det\left\{\frac{1}{[\mu^2+(x_\ell-y_m)^2][\mu^2+(x_\ell+y_m)^2]}\right.$$

$$-\frac{4m\mu^3}{\pi}\int_0^\infty dq\,\frac{q^2}{(q^2+mB)[\mu^2+(x_\ell-q)^2][\mu^2+(x_\ell+q)^2][\mu^2+(q-y_m)^2][\mu^2+(q+y_m)^2]}$$

$$= 0 \quad . \tag{IV-4}$$

We solved equation (IV-4) for λ with $m = \mu = 1$, $n = 5$ and $x_1 = y_1 = 0$, $x_2 = y_2 = 1$, ... , $x_5 = y_5 = 4$. The results are tabulated in Table I, together with the exact solution[2]

$$J_{2\sqrt{B}}(2\sqrt{\lambda}) = 0$$

where $J_x(y)$ is the Bessel function.

	1st bound state		2nd bound state		3rd bound state	
B	λe	λa	λe	λa	λe	λa
0.00	1.4458	1.468	7.6179	7.789	18.7216	19.228
0.25	3.6557	3.722	12.3061	13.045		
1.00	6.5946	6.658	17.7115	18.327		
2.25	10.1761	10.277				
4.00	14.3941	14.546				

Table 1. Value of coupling constant λ needed to produce first,second and third S-wave bound state for given binding energy B ($m=\mu=1$; λe - exact coupling constant λa - approximate coupling constant)

As one can see from Table I, Bateman's method yields results which are within 3% of the exact results. This is quite a good agreement considering the small n (=5) we chose.

IV.3 Solutions of the Schrodinger, Klein-Gordon and Our Relativistic Equations

As we have seen, equations (II-14) and (II-15) are the Schrodinger and the Klein-Gordon equation in momentum space with the Yukawa potential $v(r) = \lambda e^{-r}/r$. Now, in equation (II-14), let $m_1 = m_2 = m$ so that $2M \longrightarrow m$. Also, we shall assume that m in equation (II-15) represents the reduced mass of two interacting particles. Thus, when these two particles have equal mass, say m, then in equation (II-15) $2m \longrightarrow m$. Hence, the symmetric form of both equations (II-14) and (II-15) is, with $X_\ell(p) = p(p^2 + \omega)^{1/2} u_\ell(p)$,

$$X_\ell(p) = \lambda \frac{m}{\pi} \int_0^\infty dq \; Q_\ell\left(\frac{1+p^2+q^2}{2pq}\right) \frac{X_\ell(p)}{(p^2+\omega)^{1/2}(q^2+\omega)^{1/2}} , \tag{IV-5}$$

where $\omega = mB$ for the Schrodinger equation and $\omega = B(m-B)$

for the Klein-Gordon equation. In applying Bateman's method to equation (IV-5) we note that, for $\ell = 0$, the kernel is

$$K(x,y) = \frac{m}{2\pi} (x^2 + \omega)^{-\frac{1}{2}} (y^2 + \omega)^{-\frac{1}{2}} \ln \frac{1 + (x+y)^2}{1 + (x-y)^2} \quad .$$

Thus, $\det(A_{\ell m}) = 0$ implies that the eigenvalues λ can be obtained by solving the equation

$$\det \{ \ln \frac{1 + (x_\ell + y_m)^2}{1 + (x_\ell - y_m)^2}$$

$$- \lambda \frac{m}{2\pi} \int_0^\infty \frac{dq}{q^2 + \omega} \ln \frac{1 + (x_\ell + q)^2}{1 + (x_\ell - q)^2} \ln \frac{1 + (q + y_m)^2}{1 + (q - y_m)^2} \} = 0 \quad .$$

(IV-6)

Similarly, the symmetric form of our relativistic equation (III-22), with $m_1 = m_2 = m$ and $Z_{n\ell}(p) = p(p^2+m^2)^{-1/4} [d_{n\ell}^2 - 2d_{n\ell}(p^2+m^2)^{1/2}]^{\frac{1}{2}} f_{n\ell}(p)$, is

$$Z_{n\ell}(p) = \lambda \frac{2m^2}{\pi} \int_0^\infty dq \, Z_{n\ell}(q) \, (p^2+m^2)^{-1/4} (q^2+m^2)^{-1/4}$$

$$\cdot \; [d_{n\ell}^2 - 2d_{n\ell}(p^2+m^2)^{1/2}]^{-\frac{1}{2}} [d_{n\ell}^2 - 2d_{n\ell}(q^2+m^2)^{1/2}]^{-\frac{1}{2}}$$

$$\cdot \; Q_\ell [\frac{1 - 2m^2 + 2(p^2+m^2)^{1/2}(q^2+m^2)^{1/2}}{2pq}] \; ,$$

where $\lambda = G^2/4\pi$. Hence, for $\ell = 0$,

$$K(x,y) = \frac{m^2}{\pi}(x^2+m^2)^{-1/4}(y^2+m^2)^{-1/4}[d^2 - 2d(x^2+m^2)^{1/2}]^{-\frac{1}{2}}$$

$$\cdot [d^2 - 2d(y^2+m^2)^{1/2}]^{-\frac{1}{2}} \ln\frac{1 + (x+y)^2 - [(x^2+m^2)^{1/2} - (y^2+m^2)^{1/2}]^2}{1 + (x-y)^2 - [(x^2+m^2)^{1/2} - (y^2+m^2)^{1/2}]^2}$$

and $\det(A_{\ell m}) = 0$ implies that

$$\det \{ \ln \frac{1 + (x_\ell+y_m)^2 - [(x_\ell^2+m^2)^{1/2} - (y_m^2+m^2)^{1/2}]^2}{1 + (x_\ell-y_m)^2 - [(x_\ell^2+m^2)^{1/2} - (y_m^2+m^2)^{1/2}]^2}$$

$$- \lambda \frac{m^2}{\pi}\int_0^\infty dq(q^2+m^2)^{-1/2}[d^2 - 2d(q^2+m^2)^{1/2}]^{-1}$$

$$\cdot \quad \ln \frac{1 + (x_\ell+q)^2 - [(x_\ell^2+m^2)^{1/2} - (q^2+m^2)^{1/2}]^2}{1 + (x_\ell-q)^2 - [(x_\ell^2+m^2)^{1/2} - (q^2+m^2)^{1/2}]^2}$$

$$\cdot \quad \ln \frac{1 + (q+y_m)^2 - [(q^2+m^2)^{1/2} - (y_m^2+m^2)^{1/2}]^2}{1 + (q-y_m)^2 - [(q^2+m^2)^{1/2} - (y_m^2+m^2)^{1/2}]^2} \} = 0 \quad ,$$

(IV-7)

where $d_{00} \equiv d$ and $B_{00} \equiv B$.

To find the coupling constant λ from equations (IV-6) and (IV-7), we solved these equations numerically with $n = 10$ and $x_1 = y_1 = 0.01$, $x_2 = y_2 = 1.01$, ..., $x_{10} = y_{10} = 9.01$. The results are shown and discussed in the next section.

IV.4 Results and Discussion

We chose our system of units to be $\hbar^2 = c = \mu = 1$, where c is the speed of light and μ is the mass of the neutral meson (135.01 Mev.) . In this system of units, the mass of the neutron m (939.505 Mev.) is 6.954 pion mass, and the binding energy B is also in pion mass. Note that we are treating the neutron and the meson as scalar particles.

The results of solving equations (IV-6) and (IV-7) for λ , with $m = 6.954$ pion mass, are shown in Table 2 and Table 3 and are plotted in Fig. 1 and Fig. 2 . These results show that for weak binding the three equations agree with each other well, whereas for strong binding the difference is quite large. Note that both the Klein-Gordon and our relativistic equation predict that for a given λ we can have two possible binding energies. This can be explained by noting that in equation (IV-5), if we replace B by $m-B$ the equation remains unchanged. Thus λ is symmetric about $B = m/2$ for the Klein-Gordon equation, and for a given λ both B and $m-B$ are acceptable binding energies. Now, since our relativistic equation reduces to the Klein-Gordon equation in the limit that m_2 is

large, we expect it to retain, at least partially, this peculiar behaviour of the Klein-Gordon equation even for $m_2 = m_1$.

Table 4 shows the variation of λ with $1/m$, in the ground state, for different binding energies B . In Fig. 3 we plotted these results. From Fig. 3 we note that as m tends to infinity, the three equations agree with each other very well for all B . For m finite (but not less than one pion mass), however, the agreement is good for weak binding (B smaller than one pion mass) and very poor for strong binding (B larger than one pion mass).

Finally, to compare our results with those obtained by Schwartz and by Son and Sucher we solved our equations with $m = \mu = 1$. The results are tabulated in Table 5 and plotted in Fig. 4 .

B	λ_S	λ_{K-G}	λ_R	B	λ_S	λ_{K-G}	λ_R
0.00	0.249	0.249	0.248	3.61	1.801	1.301	1.354
0.01	0.326	0.326	0.323	4.00	1.896	1.290	1.366
0.04	0.406	0.406	0.402	4.41	1.989	1.262	1.376
0.09	0.489	0.487	0.483	4.84	2.086	1.216	1.379
0.16	0.569	0.565	0.559	5.29	2.184	1.143	1.375
0.25	0.649	0.642	0.616	5.76	2.284	1.038	1.361
0.36	0.726	0.713	0.700	6.25	2.385	0.880	1.335
0.49	0.803	0.785	0.771	6.76	2.487	0.595	1.297
0.64	0.884	0.855	0.837	7.29	2.590		1.258
0.81	0.964	0.920	0.901	7.84	2.697		1.208
1.00	1.042	0.983	0.961	8.41	2.791		1.174
1.21	1.120	1.042	1.020	9.00	2.901		1.062
1.44	1.205	1.089	1.078	9.61	3.029		0.968
1.69	1.287	1.148	1.124	10.24	3.144		0.832
1.96	1.372	1.194	1.179	10.89	3.260		0.732
2.25	1.455	1.234	1.224	11.56	3.378		0.592
2.56	1.541	1.264	1.262	12.25	3.498		0.460
2.89	1.631	1.287	1.297	12.96	3.622		0.269
3.24	1.728	1.299	1.331	13.69	3.747		

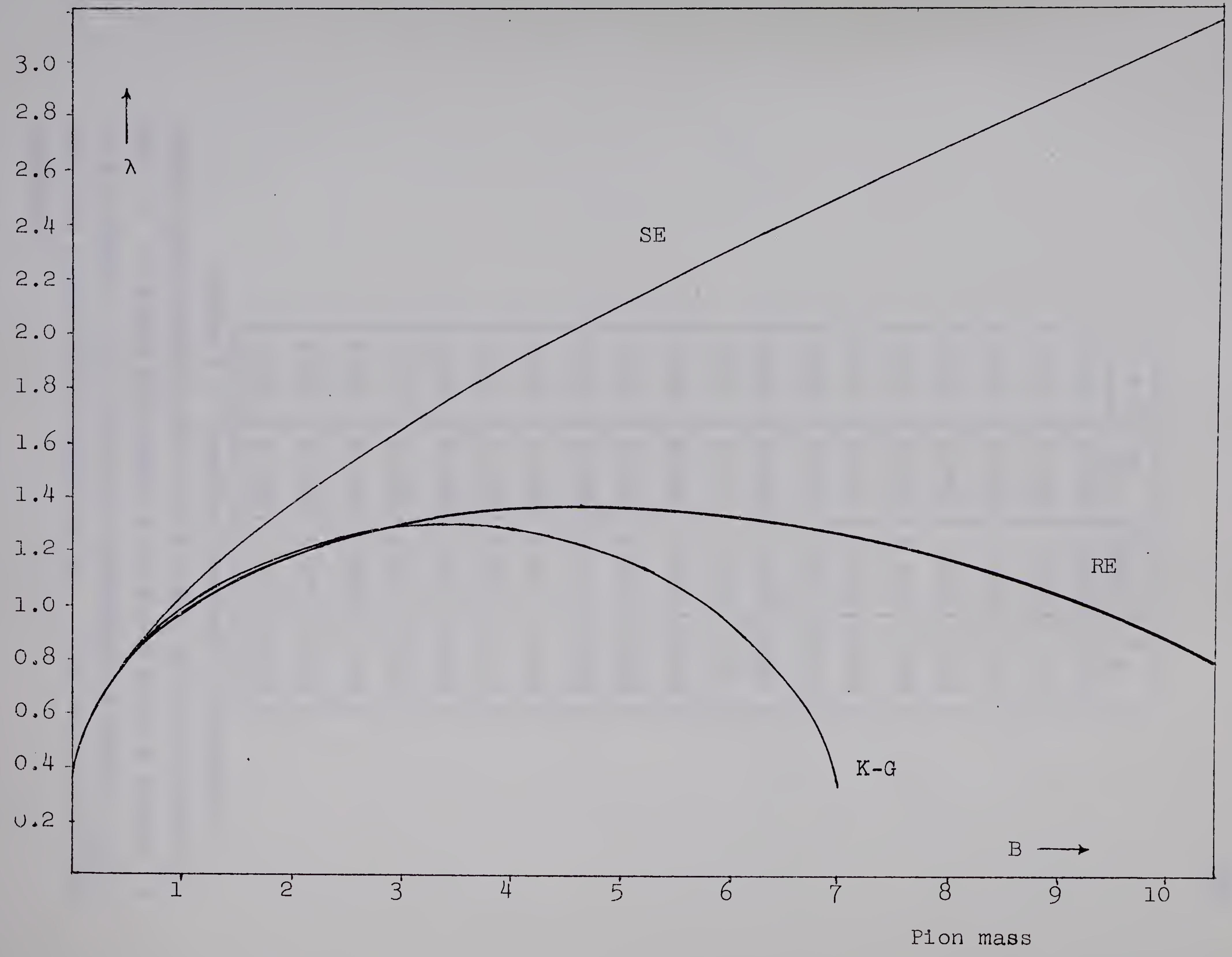
Table 2. Coupling constant λ required to produce the first S-wave bound state of two neutrons of mass m (= 6.954 pion mass) with given binding energy B in pion mass. (λ_S - from Schrodinger equation; λ_{K-G} - from the Klein-Gordon equation; λ_R - from our relativistic equation)

Fig. 1. Ground state coupling constant λ versus the binding energy B in pion mass unit.

SE - from the Schrodinger equation

RE - from our relativistic equation

K-G - from the Klein-Gordon equation


λ
SE
RE
K-G
B
Pion mass
3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7
8
9
10

B	λ_S	λ_{K-G}	λ_R
0.00	0.944	0.944	0.955
0.01	1.132	1.132	1.135
0.04	1.336	1.334	1.359
0.09	1.485	1.482	1.514
0.16	1.643	1.634	1.678
0.25	1.852	1.835	1.804
0.36	2.041	2.010	2.052
0.49	2.239	2.189	2.261
0.81	2.651	2.536	2.640
1.00	2.896	2.703	2.874
1.21	3.131	2.895	3.096
1.44	3.346	3.101	3.283
1.96	3.791	3.315	3.536
2.25	4.094	3.425	3.716
3.24	4.898	3.537	4.242
4.00	5.366	3.507	4.456
4.41	5.749	3.438	4.542
4.84	6.043	3.377	4.597
5.29	6.388	3.179	4.623

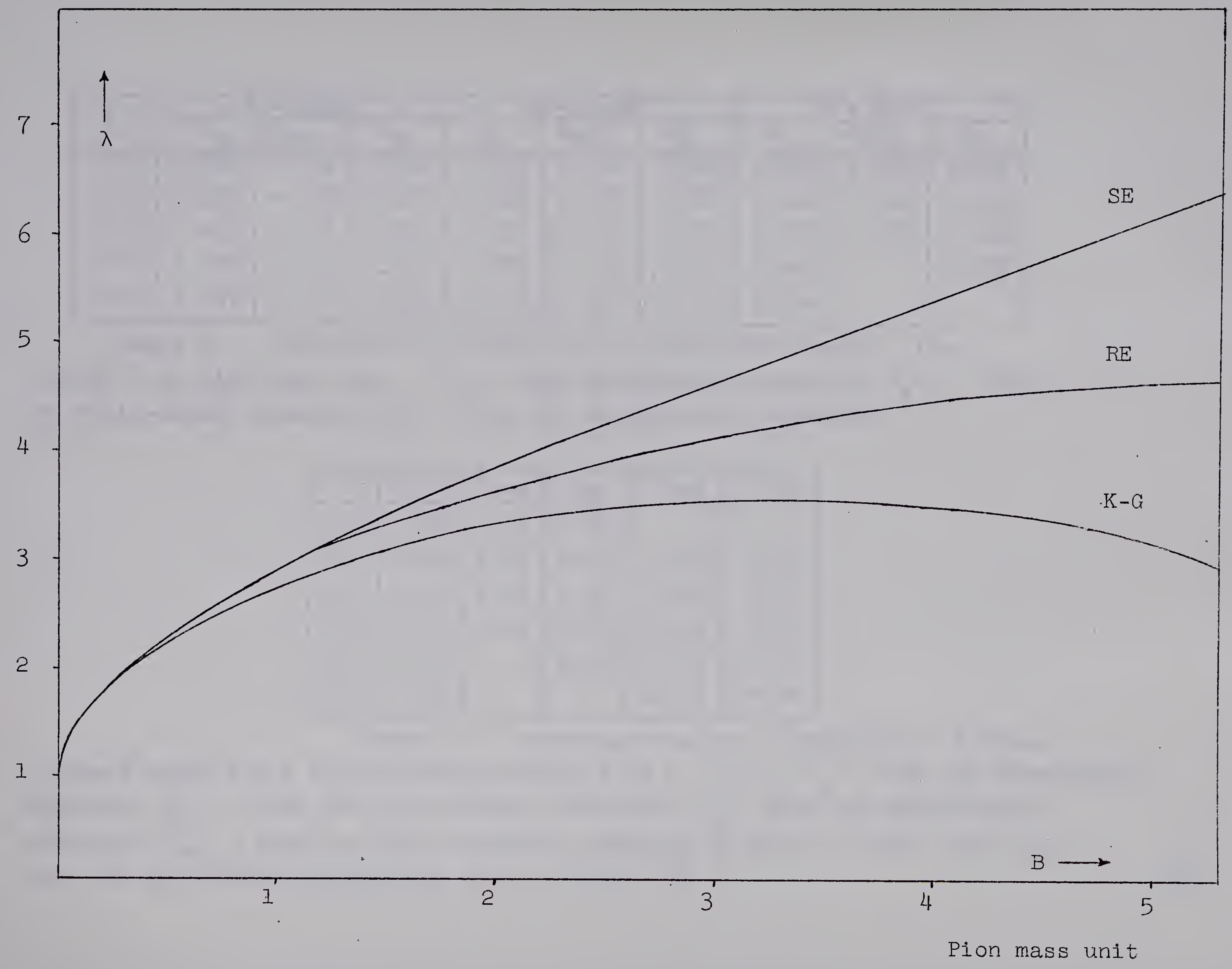
Table 3. Coupling constant λ required to produce the second S-wave bound state of two neutrons of mass m (= 6.954 pion mass) with given binding energy B in pion mass. (λ_S - from the Schrodinger equation; λ_{K-G} - from the Klein-Gordon equation; λ_R - from our relativistic equation.)

Fig. 2. First excited state coupling constant λ versus the binding energy B in pion mass unit.

SE - from the Schrodinger equation

RE - from our relativistic equation

K-G - from the Klein-Gordon equation


λ
SE
RE
K-G
B
Pion mass unit
1
2
3
4
5
6
7

$\frac{1}{m}$	B = 0.36			B = 1.69			B = 9.00		
	λ_S	λ_{K-G}	λ_R	λ_S	λ_{K-G}	λ_R	λ_S	λ_{K-G}	λ_R
0.01	0.082	0.082	0.082	0.267	0.266	0.266	1.269	1.263	1.261
0.10	0.188	0.188	0.188	0.432	0.425	0.427	1.519	1.352	1.423
0.50	0.465	0.461	0.460	0.877	0.827	0.827	2.235	1.203	1.533
1.00	0.726	0.713	0.700	1.287	1.148	1.124	2.901		1.062
4.00	1.947	1.843	1.690	3.055	1.321	1.555	5.814		

Table 4. Variation of λ with 1/m in $(6.954 \text{ pion mass})^{-1}$ for given B in pion mass unit. (λ_S - from Schrodinger equation; λ_{K-G} - from the Klein-Gordon equation; λ_R - from our relativistic equation)

B	λ_S	λ_{K-G}	λ_R	λ_{B-S}	λ_{S-S}
0.0	1.68	1.70	1.60		
0.2	2.64	2.55	2.28	5.25	5.03
0.4	3.06	2.72	2.34	6.60	6.47
0.8	3.60	2.55	2.10	8.54	8.86
1.2	4.00		1.73	9.79	11.00
2.0	4.69			10.74	14.90

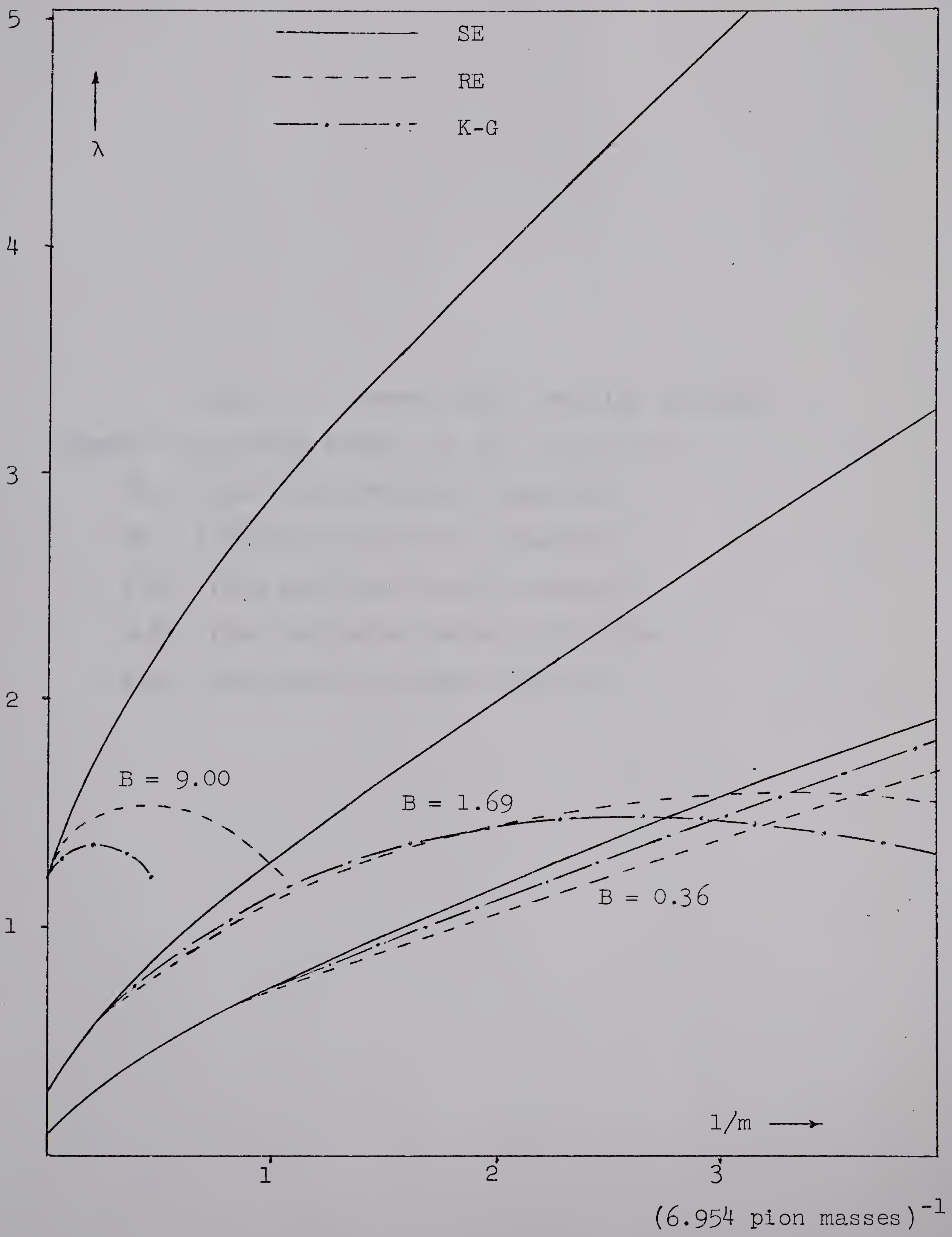
Table 5. Coupling constant λ required to produce ground state for a given binding energy B ($m = \mu = 1$; λ_S - from the Schrodinger equation; λ_{K-G} - from the Klein-Gordon equation; λ_R - from our relativistic equation; λ_{B-S} - from the Bethe-Salpeter equation as given in ref. (4); λ_{S-S} - from the Son-Sucher equation as given in ref. (5)).

Fig. 3. Variation of λ with the inverse of the mass, $1/m$, in (6.954 pion mass)$^{-1}$ for given binding energy B in pion mass unit.

SE - from the Schrodinger equation

K-G - from the Klein-Gordon equation

RE - from our relativistic equation



SE
RE
K-G
λ
B = 9.00
B = 1.69
B = 0.36
1/m
5
4
3
2
1
1
2
3
(6.954 pion masses)$^{-1}$

Fig. 4. Ground state coupling constant λ versus the binding energy B for $m = \mu = 1$.

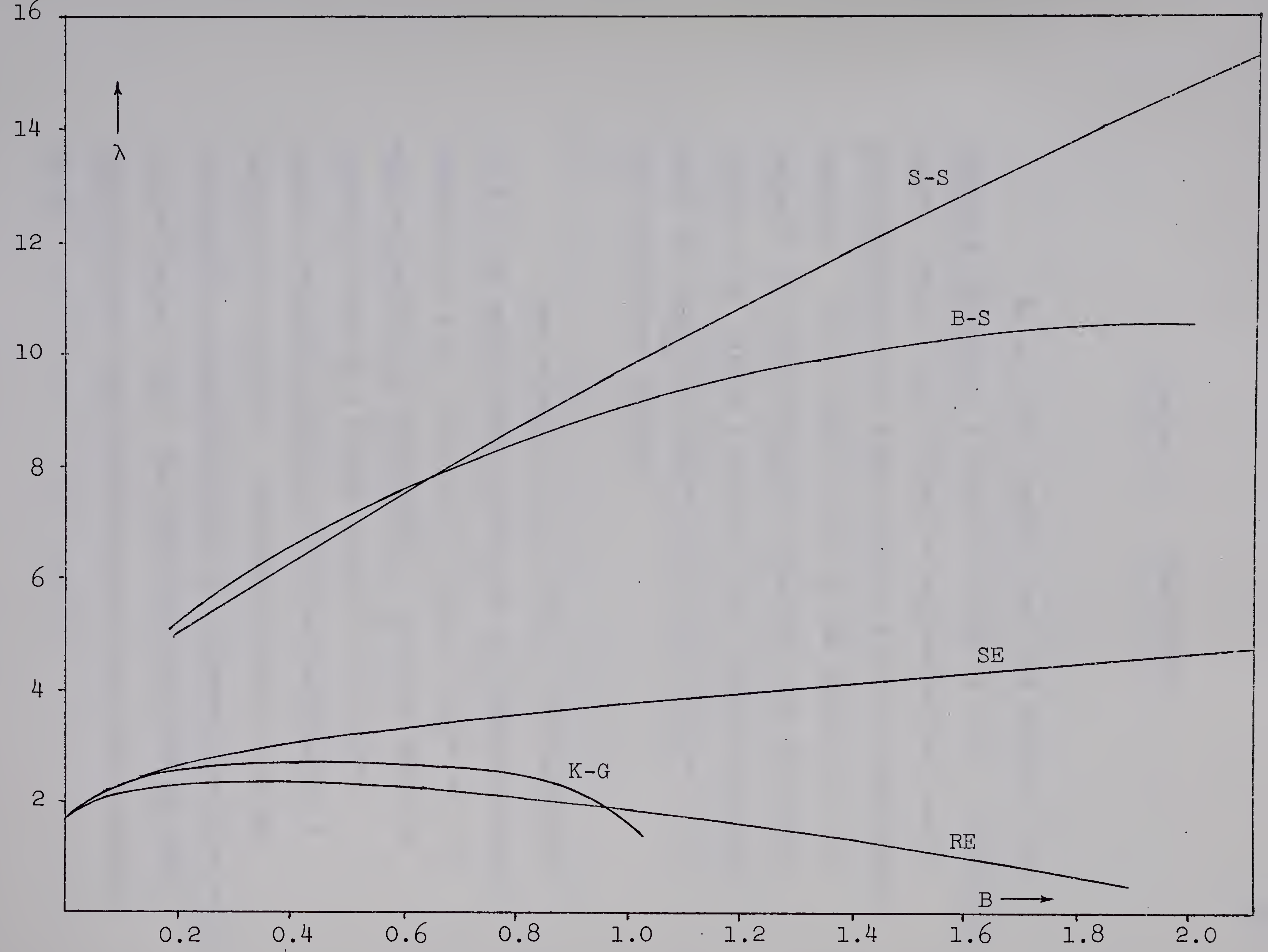
SE - from the Schrodinger equation

RE - from our relativistic equation

K-G - from the Klein-Gordon equation

B-S - from the Bethe-Salpeter equation

S-S - from the Son-Sucher equation


16
14
12
10
8
6
4
2
λ
S-S
B-S
SE
K-G
RE
B
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Chapter V. CONCLUSIONS

As we have already seen, the values of the coupling constants, for weak binding ($B \ll m$) , predicted by the lowest-order "N-quantum approximation" are in good agreement with those predicted by the Schrodinger equation. On the other hand, as was pointed out by Vosko[10], the Bethe-Salpeter equation, for very small values of B ,yields values for the coupling constants which are about 40% larger than those obtained from the Schrodinger equation.

For strong binding ($B \gtrsim m$), the agreement of all the relativistic equations with the Schrodinger equation is poor. This is not surprising since, as was pointed out by Greeberg[11], a particle which is strongly bound by a Yukawa potential must move relativistically and thus, must be described by a relativistic equation. Moreover, since all of the relativistic equations reduce to a Yukawa-like interaction in the proper limit, the particles described by these equations must move relativistically and independently of m .

To see if one can represent the interaction predicted by the "N-quantum approximation", in terms of a potential, we write both the Schrodinger and the Klein-Gordon equations in the most general (non-local) form as (see Appendix)

$$\left(\frac{p^2}{m} + \omega \right) u(\vec{p}) = - \int d^3q \; v(\vec{p},\vec{q}) \; u(\vec{q}) \quad ,$$

where $\omega = B$ for the Schrodinger equation and $\omega = B(m-B)/m$ for the Klein-Gordon equation. We also rewrite our relativistic equation (III-21) as

$$\left(\frac{p^2}{m} + \omega\right) f(\vec{p}) = - \frac{2G^2m^2}{(2\pi)^3} \int d^3q \; \frac{(p^2/m + \omega) f(\vec{q})}{q_0(d^2-2dp_0)(1-2m^2+2p_0q_0-2\vec{p}\cdot\vec{q})} ,$$

where $p_0 = (p^2+m^2)^{1/2}$ and $q_0 = (q^2+m^2)^{1/2}$.

Comparing the above two equations we get

$$v(\vec{r},\vec{r}\,') = (2\pi)^{-3} \iint d^3p d^3q \; e^{-i\vec{p}\cdot\vec{r}} v(\vec{p},\vec{q}) e^{i\vec{q}\cdot\vec{r}\,'}$$

$$= \frac{G^2m^2}{4\pi^4} (2\pi)^{-3} \iint d^3p d^3q \; \frac{e^{-i\vec{p}\cdot\vec{r}} (p^2/m + \omega) e^{i\vec{q}\cdot\vec{r}\,'}}{q_0(d^2-2dp_0)(1-2m^2+2p_0q_0-2\vec{p}\cdot\vec{q})} \quad .$$

To approximate the above integral we expanded the kernel, $v(\vec{p},\vec{q})$, in terms of m (see Appendix). Thus, up to $1/m^3$

we have

$$\int v(\vec{r},\vec{r}\,')\Psi(\vec{r}\,')d^3r' = -\frac{G^2}{4\pi}\left(1 + \frac{B}{2m} + \frac{B^2}{4m^2} + \frac{B^3}{8m^3}\right)\frac{e^{-r}}{r}\Psi(\vec{r}) + F_1\Psi(\vec{r})$$

for the Schrodinger equation, and

$$\int v(\vec{r},\vec{r}\,')\Psi(\vec{r}\,')d^3r' = -\frac{G^2}{4\pi}\left(1 - \frac{B}{2m} - \frac{B^2}{4m^2} - \frac{B^3}{8m^3}\right)\frac{e^{-r}}{r}\Psi(\vec{r}) + F_2\Psi(\vec{r})$$

for the Klein-Gordon equation, where (see Appendix) F_1 and F_2 are complicated functions of B, m, $\vec{r}$, and $\vec{p}$ $(= -i\nabla)$. Thus, we see that the lowest-order N-quantum approximation predicts a non-local energy-dependent interaction. For m very large the non-locality tends to zero and up to order of $1/m$ we have

$$v(\vec{r},\vec{r}\,') = -\frac{G^2}{4\pi}\left(1 + \frac{B}{2m}\right)\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')$$

for the Schrodinger equation and

$$v(\vec{r},\vec{r}\,') = -\frac{G^2}{4\pi}\left(1 - \frac{B}{2m}\right)\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')$$

for the Klein-Gordon equation. Hence, $1/m$ roughly represents the range of the non-locality of the interaction.

Finally, we note that one can use equation (III-21) and Bateman's method to determine numerically the wave functions of the bound states. This in turn would help us to understand, at least qualitatively, the overall nature of the interaction (repulsive or attractive) and provide an interesting comparison to the wave functions one obtains from the Bethe-Salpeter equation[12,13] .

BIBLIOGRAPHY

1. See for example L. I. Schiff "Quantum Mechanics", McGraw-Hill Inc. (1955) Section 16

2. See for example H. A. Bethe and R. F. Bacher Rev. Mod. Phys. 8, 82 (1936)

3. H. Yukawa, Proc. Math. Soc. Japan 17, 48 (1935); Reprint from "Nuclear Forces" by D. M. Brink, Pergamon Press (1965) p. 214

4. Charles Schwartz, Phys. Rev. 137, B717 (1965)

5. Nguyen D. Son and J. Sucher, Phys. Rev. 153, 1496 (1967)

6. O. W. Greenberg, Phys. Rev. 139, B1038 (1965)

7. O. W. Greenberg and R. J. Genolio, Phys. Rev. 150, 1076 (1966)

8. R. Haag, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 29, No. 12 (1955)

9. H. Bateman, Roy. Soc. Proc. 100, 441 (1922); L. V. Kantorovich and V. I. Krylov "Approximate Methods of Higher Analysis", P. Noordhoff Ltd. p. 155 (1958)

10. S. H. Vosko, J. Math. Phys. 1, 505 (1960)

11. O. W. Greenberg, Phys. Rev. 147, 1077 (1966)

12. M. Kawaguchi, Prog. Theor. Phys. 33, 932 (1965)

13. P. J. Roberts, Nuc. Phys. B1, 115 (1967)

APPENDIX

THE FORM OF THE INTERACTION PREDICTED BY OUR RELATIVISTIC EQUATION

The Schrodinger and the Klein-Gordon equations with the non-local potential $v(\vec{r},\vec{r}\,')$ can be written as

$$\left(-\frac{\nabla^2}{m}+\omega\right)\Psi(\vec{r}) = -\int d^3r'\; v(\vec{r},\vec{r}\,')\Psi(\vec{r}\,') \tag{A-1}$$

where $\omega = B$ for the Schrodinger equation and $\omega = B(m-B)/m$ for the Klein-Gordon equation. Let

$$\begin{aligned} v(\vec{r},\vec{r}\,') &= (2\pi)^{-3}\iint d^3p\,d^3q\; e^{-i\vec{p}.\vec{r}}\; v(\vec{p},\vec{q})\; e^{i\vec{q}.\vec{r}\,'} \qquad \text{(A-2)} \\ &\equiv T[v(\vec{p},\vec{q})] \; . \end{aligned}$$

Then,

$$\begin{aligned} v(\vec{p},\vec{q}) &= (2\pi)^{-3}\iint d^3r\,d^3r'\; e^{i\vec{p}.\vec{r}}\; v(\vec{r},\vec{r}\,')\; e^{-i\vec{q}.\vec{r}\,'} \qquad \text{(A-3)} \\ &\equiv T^{-1}[v(\vec{r},\vec{r}\,')] \; . \end{aligned}$$

Also let

$$\Psi(\vec{r}) = (2\pi)^{-3/2} \int d^3p \; u(\vec{p}) \; e^{-i\vec{p}.\vec{r}} \quad . \tag{A-4}$$

Substituting equations (A-4) and (A-2) into equation (A-1) we get

$$\left(\frac{p^2}{m} + \omega \right) u(\vec{p}) = - \int d^3q \; v(\vec{p},\vec{q}) \; u(\vec{q}) \quad . \tag{A-5}$$

Consider now equation (III-21) with $m_1 = m_2$

$$(d^2 - 2dp_0) f(\vec{p}) = - \frac{2G^2m^2}{(2\pi)^3} \int d^3q \; \frac{f(\vec{q})}{q_0(1 - 2m^2 + 2p_0q_0 - 2\vec{p}.\vec{q})} \quad ,$$

where $d = 2m - B$, $p_0 = (p^2 + m^2)^{1/2}$, $q_0 = (q^2 + m^2)^{1/2}$ and $f_{n\ell}(\vec{p}) = f(\vec{p})$. This equation can be rewritten as

$$\left(\frac{p^2}{m} + \omega\right) f(\vec{p}) = - \frac{2G^2m^2}{(2\pi)^3} \int d^3q \; \frac{(p^2/m + \omega) f(\vec{q})}{q_0(d^2-2dp_0)(1-2m^2+2p_0q_0-2\vec{p}.\vec{q})} \quad . \tag{A-6}$$

Comparing equation (A-6) with equation (A-5) we see that

$$v(\vec{p},\vec{q}) = \frac{G^2}{4\pi} \frac{m^2}{\pi^2} \frac{(p^2/m + \omega)}{q_0(d^2-2dp_0)(1-2m^2+2p_0q_0-2\vec{p}.\vec{q})} \tag{A-7}$$

and using equation (A-2) we have

$$v(\vec{r},\vec{r}\,') = \frac{G^2}{4\pi}\frac{m^2}{\pi^2}(2\pi)^{-3}\iint d^3p\,d^3q\;\frac{e^{-i\vec{p}.\vec{r}}\;(p^2/m+\omega)\;e^{i\vec{q}.\vec{r}\,'}}{q_0(d^2-2dp_0)(1-2m^2+2p_0q_0-2\vec{p}.\vec{q})}\,. \tag{A-8}$$

We proceed to approximate equation (A-8) by assuming that m is large and thus expanding the kernel, $v(\vec{p},\vec{q})$, in powers of $1/m$. Hence,

$$1/q_0 = (q^2+m^2)^{1/2} \approx \frac{1}{m}\left[\,1+\frac{a_1}{m^2}+\frac{a_2}{m^4}\,\right] ,$$

where $a_1 = -\,q^2/2$ and $a_2 = 3q^4/8$;

$$[1-2m^2+2(p^2+m^2)^{1/2}(q^2+m^2)^{1/2}-2\vec{p}.\vec{q}]^{-1}$$

$$\approx [1+(\vec{p}-\vec{q})^2]^{-1}\left[1+\frac{b_1}{m^2}+\frac{b_2}{m^4}\,\right] ,$$

where

$$b_1 = \frac{(p^2-q^2)^2}{4[1+(\vec{p}-\vec{q})^2]} \quad\text{and}\quad b_2 = -\,\frac{(p^2-q^2)^2}{8[1+(\vec{p}-\vec{q})^2]}\left\{p^2+q^2-\frac{(p^2-q^2)^2}{2[1+(\vec{p}-\vec{q})^2]}\right\}$$

and

$$(d^2-2dp_0)^{-1} = (2m-B)^{-1}[2m-B-2(p^2+m^2)^{1/2}]^{-1}$$

$$\approx -(2mB)^{-1}[1 + \frac{c_1}{m} + \frac{c_2}{m^2} + \frac{c_3}{m^3} + \frac{c_4}{m^4}] \quad ,$$

where $c_1 = -p^2/B + B/2$, $c_2 = B^2/4 + p^4/B^2 - p^2/2$,
$c_3 = B^3/8 + p^4/2B + p^4/4B - Bp^2/4 - p^6/B^3$ and
$c_4 = B^4/16 - p^6/2B^2 + p^8/B^4 + p^4/8 - p^6/2B^2 + p^4/4 - B^2p^2/8$.
Hence, up to $1/m^3$ $v(\vec{p},\vec{q})$ can be written as

$$v(\vec{p},\vec{q}) = -\frac{G^2}{4\pi}\frac{1}{\pi^2}\frac{1}{2B}\frac{1}{1+(\vec{p}-\vec{q})^2}[v_0 + \frac{v_1}{m} + \frac{v_2}{m^2} + \frac{v_3}{m^3}] \quad ,$$

where $v_0 = B$, $v_1 = B^2/2$, $v_2 = B^3/4 + \frac{B}{4}[\frac{(p^2-q^2)^2}{1+(\vec{p}-\vec{q})^2} - 2q^2]$

and $v_3 = \frac{p^4}{4} + \frac{B^4}{8} + \frac{B^2}{8}[\frac{(p^2-q^2)^2}{1+(\vec{p}-\vec{q})^2} - 2q^2]$ for the
Schrodinger equation and
$v_0 = B$, $v_1 = -B^2/2$, $v_2 = -\frac{B^3}{4} + \frac{B}{4}[\frac{(p^2-q^2)^2}{1+(\vec{p}-\vec{q})^2} - 2q^2 + 4p^2]$,

and $v_3 = -\frac{B^4}{8} - \frac{3p^4}{4} - \frac{B^2}{8}[\frac{(p^2-q^2)^2}{1+(\vec{p}-\vec{q})^2} - 2q^2]$ for the

Klein-Gordon equation. Now, using the identities

$$T\,[\frac{1}{2\pi^2}\frac{1}{1+(\vec{p}-\vec{q})^2}] = \frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')$$

and

$$T\,[\frac{1}{\pi^2}\frac{1}{[1+(\vec{p}-\vec{q})^2]^2}] = e^{-r}\delta(\vec{r}-\vec{r}\,')$$

we get

$$v(\vec{r},\vec{r}\,') = -\frac{G^2}{4\pi}\{\ (1 + \frac{B}{2m} + \frac{B^2}{4m^2} + \frac{B^3}{8m^3})\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')$$

$$+ \frac{1}{2m^2}(1 + \frac{B}{2m})\frac{e^{-r}}{r}\ \nabla^2_{r'}[\delta(\vec{r}-\vec{r}\,')] - \frac{B}{4m^3}\ \nabla^4_r[\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')]$$

$$+ \frac{1}{4m^2}(1 + \frac{B}{2m})(\nabla^2_r - \nabla^2_{r'})^2[e^{-r}\delta(\vec{r}-\vec{r}\,')]\ \}$$

for the Schrodinger equation and .

$$v(\vec{r},\vec{r}\ ) = -\frac{G^2}{4\pi}\{\ (1 - \frac{B}{2m} - \frac{B^2}{4m^2} - \frac{B^3}{8m^3})\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')$$

$$+ \frac{1}{8m^2}(1 - \frac{B}{2m})(\nabla^2_r - \nabla^2_{r'})^2[e^{-r}\delta(\vec{r}-\vec{r}\,')] + \frac{1}{2m^2}(1 - \frac{B}{2m})\frac{e^{-r}}{r}\nabla^2_{r'}[\delta(\vec{r}-\vec{r}\,')]$$

$$+ (\frac{3}{4Bm^3}\ \nabla^4_r - \frac{1}{m^2}\ \nabla^2_r)[\frac{e^{-r}}{r}\delta(\vec{r}-\vec{r}\,')]\ \}$$

for the Klein-Gordon equation with $\nabla_r = \frac{\partial}{\partial\vec{r}}$ and $\nabla_{r'} = \frac{\partial}{\partial\vec{r}\,'}$.

Thus, in equation (A-1)

$$\int d^3r'\,v(\vec{r},\vec{r}\,')\Psi(\vec{r}\,') \approx -\frac{G^2}{4\pi}(1 + \frac{B}{2m} + \frac{B^2}{4m^2} + \frac{B^3}{8m^3})\frac{e^{-r}}{r}\Psi(\vec{r}) + F_1\Psi(\vec{r})\ ,$$

for the Schrodinger equation, and

$$\int d^3r' v(\vec{r},\vec{r}')\Psi(\vec{r}') \approx -\frac{G^2}{4\pi}\left(1 - \frac{B}{2m} - \frac{B^2}{4m^2} - \frac{B^3}{8m^3}\right)\frac{e^{-r}}{r}\Psi(\vec{r}) + F_2\Psi(\vec{r})$$

for the Klein-Gordon equation where

$$F_1 = \frac{1}{2m^2}\left(1+\frac{B}{2m}\right)\frac{e^{-r}}{r}\nabla^2 - \frac{B}{4m^3}\nabla^4\left[\frac{e^{-r}}{r}\right] + \frac{1}{8m^2}\left(1+\frac{B}{2m}\right)\left(\nabla^4 e^{-r} - 2\nabla^2 e^{-r}\nabla^2 + e^{-r}\nabla^4\right)$$

and

$$F_2 = \frac{1}{2m^2}\left(1-\frac{B}{2m}\right)\frac{e^{-r}}{r}\nabla^2 - \frac{3}{4m^3}\nabla^4\left[\frac{e^{-r}}{r}\right] + \frac{1}{8m^2}\left(1-\frac{B}{2m}\right)\left(\nabla^4 e^{-r} - 2\nabla^2 e^{-r}\nabla^2 + e^{-r}\nabla^4\right)$$

and thus we see that the interaction is non-local and energy-dependent.